M. Math. IInd year Backpaper Exam 2022
Commutative Algebra
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Attempt ONLY FOUR problems.
Each Question carries 12 marks; a score of 45 or more will be taken as 45.
Unless specified otherwise, all rings are commutative with unity.

Q 1a. Show that every finitely presented, flat A-module is projective.

OR

Q 1b. If A is a domain in which each finitely generated ideal is principal, show that a module is flat if and only if it is torsion-free.

Q 2a. Let $0 \to L \to M \to N \to 0$ be a short exact sequence of A-modules where M is finitely generated and N is finitely presented. Prove that L must be finitely generated.

OR

Q 2b. For ideals I, J show that the A-modules $Tor_1(A/I, A/J)$ and $Tor_2(A/I, A/J)$ are isomorphic to $(I \cap J)/IJ$ and $Ker(I \otimes J \to IJ)$ respectively.

Q 3a. Show that if Spec(A) is not connected, then $A \cong A_1 \times A_2$ where the rings A_1, A_2 are both non-zero.

OR

Q 3b. For local subrings A, B of a field K, recall that A is said to be dominated by B if A is a subring of B and the maximal ideal of A is the contraction of the maximal ideal of B. Prove that any valuation ring C is maximal with respect to the partial order induced by dominance for local subrings of the quotient field of C.

Q 4a. Let G be a finite group of automorphisms of a ring A. Prove that A is an integral extension of $A^G := \{a \in A : g(a) = a\}$.

OR

Q 4b. If all prime ideals of a ring A are principal, prove that all ideals must be principal.

Q 5a. If $I \subset \mathbb{R}[X_1, \cdots, X_n]$ is an ideal and $V_{\mathbb{R}}(I) := \{x \in \mathbb{R}^n : f(x) = 0 \ \forall \ x \in I\}$, then observe

$$V_{\mathbb{R}}(I) = \{x \in \mathbb{R}^n : (f_1^2 + f_2^2 + \cdots + f_d^2)(x) = 0\}$$

where $I = (f_1, \cdots, f_d)$.
In general, for any field K which is not algebraically closed, prove the analogous statement that the set of zeroes of a family of polynomials in $K[X_1, \cdots, X_n]$ is the zero set of a single polynomial.

OR

Q 5b. If $f : M \to N$ is an A-module homomorphism such that the induced homomorphisms $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is injective for each maximal ideal $\mathfrak{m}$ of A, prove that f must be injective.